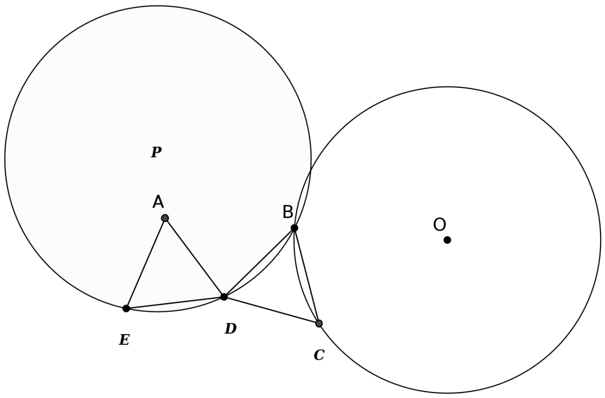
Monthly Mathematics Challenge Solutions – April 2016

1. The triangles in the figure are equilateral. The circles have the same radius. The point O is the center of the circle on the right. Show that the points A, B, and O are on a line.



Solution: Refer to the annotated figure above for point labels. Note that P is the center of the circle containing it.

Let $\angle PDB = \theta$. Because $\triangle PED$ is congruent to $\triangle PDB$, $\angle PDB = \angle PDE$. Thus $\angle EDB = 2\theta$. Since $\triangle ADE$ is equilateral, $\angle ADE = 60°$ and $\angle ADB = \angle EDB - \angle ADE = 2\theta - 60°$. Since $\triangle ADB$ is isoceles, $\angle ABD = [180° - (2\theta - 60°)]/2 = 120° - \theta$. Since $\triangle BDC$ is equilateral, $\angle DBC = 60°$. Since the circles have the same radius, $\triangle PDB$ is congruent to $\triangle OBC$ and it follows that $\angle OBC = \theta$. Hence, $\angle ABD + \angle DBC + \angle CBO = 120° - \theta + 60° + \theta = 180°$, and the points A, B, and O are on a line.

2. Let

$$\sum_{n=1}^{\infty} a_n$$

be a series of real numbers that is conditionally convergent but not absolutely convergent. Show that the numbers $a_1, a_2, \ldots$ can be arranged in an infinite array such that every row sum is zero and every column sum is zero.

Solution: We first prove a lemma.

Lemma. *Given the first n terms of a sequence, we can form the rest of the sequence by choosing terms from the conditionally convergent series such that the sequence sums to zero.*

Let a conditionally convergent series and the first n terms of a sequence be given. Let K_0 denote the sum of the n terms. Let P denote the set of positive terms in the conditionally convergent series and let N denote the negative terms. Without loss of generality suppose that $K_0 > 0$. We can choose terms from N so that when they are added to K the new sum K_1 is just below zero, then we can choose terms from P so that when they are added to K_1 the result K_2 is just above zero. This process can be iterated to create a sequence which sums to zero, since the terms of the original sequence converge to zero.

Appealing to this lemma we can choose a sequence of values for the first row of the array, then a sequence of values for the first column. Continuing in this fashion, second row then second column, and so on, we can construct the desired infinite array.

3. Three balanced dice are rolled. If no two dice show the same face, what is the probability that one is an ace? Note: rolling an ace means rolling a one. Extend this result to cover the cases $n = 1, 2, \ldots, 6$. That is, find p_n the conditional probability that an ace is observed given that no two dice show the same face, when n dice are rolled.

Solution: With three dice there are $6 \cdot 5 \cdot 4 = 120$ possibilities in which the three faces are all different. Among these there are $1 \cdot 5 \cdot 4 = 20$ triples with the first element being an ace, $5 \cdot 1 \cdot 4 = 20$ with the second being an ace, and $5 \cdot 4 \cdot 1 = 20$ with the third being an ace. Thus, 60 of the 120 possibilities in which the three faces are all different contain one ace and the probability one of three dice is an ace given that no two dice show the same face is $3/6 = 1/2$.

This argument is readily extended to cover the cases $k = 1, 2, 3, 4, 5, 6$. When k dice are tossed, assuming that no two dice show the same face,

$$P(1 \text{ ace}) = \frac{\binom{k}{1} \cdot 5 \cdot 4 \cdots (6-k+1)}{6 \cdot 5 \cdot 4 \cdots (6-k+1)} = \frac{k}{6}.$$