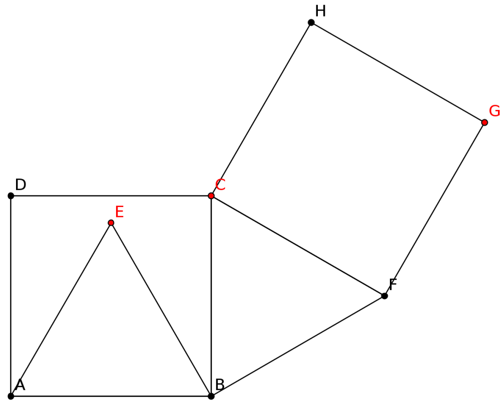
Monthly Mathematics Challenge Solutions – February 2016

1. The squares and triangles in the figure all have sides of one unit length. Show that the points E, C, and G are on a line and find the distance from E to G.



Solution: Triangle $\triangle EBC$ is isoceles, since the line segments EB and BC each have unit length. Triangle $\triangle AEB$ is equilateral, since EB, AB, and AE each have unit length. Therefore, $\angle EBA = 60°$, which implies that $\angle EBC = 30°$. Together with the fact that $\triangle EBC$ is isoceles, this implies that $\angle ECB = 75°$.

Triangle $\triangle CBF$ is equilateral, since CB, BF, and CF each have unit length. Therefore, $\angle BCF = 60°$.

Since the diagonals of a square bisect the vertex angles, $\angle GCF = 45°$.

Combining these results we find that $\angle ECB + \angle BCF + \angle GCF = 75° + 60° + 45° = 180°$. This implies that E, C, and G lie on a line.

Since $\triangle EBC$ is isoceles, $\angle EBC = 30°$, and sides EB and BC each have unit length, the distance from E to C is $\sqrt{2 - 2\cos(30)} = \sqrt{2(1 - \cos(30))} = \sqrt{4\sin^2(15)} = 2\sin(15)$. The line segment CG has length $\sqrt{2}$, since it is a diagonal of a unit square. Hence, the distance from E to G is $\sqrt{2} + 2\sin(15)$.

2. Find the largest number a such that $f(x) = \sin(x) - x + ax^3$ is an increasing function.

Solution: We will show that this claim is false by proving that there is no upper bound on the values of a for which $f(x)$ is increasing.

The derivative of $f(x) = \sin(x) - x + ax^3$ is $f'(x) = \cos(x) - 1 + 3ax^2$. Thus, $f(x)$ is increasing when $\cos(x) - 1 + 3ax^2 > 0$ or equivalently when $a > \frac{1-\cos(x)}{3x^2}$. We will first show that $\frac{1-\cos(x)}{3x^2}$ is bounded above. This follows from:

(1) For $|x| > 1$, $\frac{1-\cos(x)}{3x^2} \leq 1/3$.

(2) For $x \neq 0$, $\frac{1-\cos(x)}{3x^2}$ is continuous.

(3) For $x = 0$, two applications of L'Hospital's Rule gives $\lim_{x\to\infty} \frac{1-\cos(x)}{3x^2} = 1/6$.

Since $\frac{1-\cos(x)}{3x^2}$ is bounded above, we can choose a as large as we want provided it is greater than the upper bound.

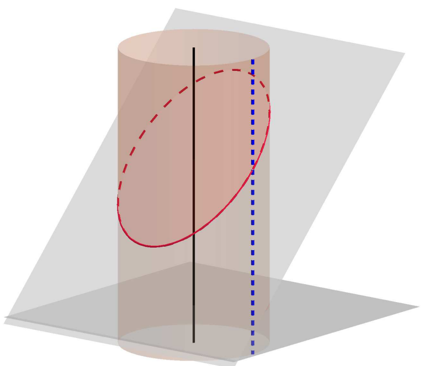
3. Show that $\sqrt{2} + \sqrt[3]{3}$ is an irrational number.

Solution: Let $a = \sqrt{2} + \sqrt[3]{3}$. Then, $a - \sqrt{2} = \sqrt[3]{3}$ and cubing both side yields

$$(a^3 + 6a) + (2^{3/2} - 3a^2\sqrt{2}) = 3 \tag{1}$$

Assume that a is rational, then $a^3 + 6a$ is rational. Since $\sqrt{2}$ is irrational and the product of of a rational and an irrational is irrational, $3a^2\sqrt{2}$ is irrational. Thus $2^{3/2} - 3a^2\sqrt{2}$ is irrational, since $2^{3/2}$ is irrational. This shows that the left hand side of equation (1) is irrational, since it is expressed as the sum of a rational and an irrational. Because 3 is rational, this leads to a contradiction and establishes the fact that a is irrational.

4. A plane cuts a paper cylinder at an angle of 45° with respect to the cylinder's axis. The intersection of the plane and the cylinder is the curve shown in red in the first figure. The cylinder is now cut along a line parallel to its axis and opened up on a flat plane. Show that the red curve becomes a whole period of a sine curve.



Brief Solution: Without loss of generality assume that the cylinder has radius one. Take the point where the axis intersects the plane which cuts the cylinder to be the origin. Because the plane cuts the cylinder at a 45° angle, the cylindrical coordinates of the curve are of the form $(1, \theta, \sin(\theta))$. Thus, as such a point is moved around the cylinder on the plane through the origin and perpendicular to the cylinder axis the angle θ varies through a 360° arc and the height coordinate $\sin(\theta)$ traces a whole period of a sine curve.

Detailed Solution: Without loss of generality assume that the cylinder has radius one. Consider a reference plane perpendicular to the axis of the cylinder which intersects the cylinder axis at the same point where the angled plane does. Take the point where the cylinder axis intersects these planes to be the origin and the line where the planes intersect as the reference axis. Define cylindrical coordinates of the form (r, θ, h), where r is a radius (distance on the reference axis), θ is an angle from the reference axis, and h is a height perpendicular to the reference plane. Points on the intersection of the cylinder and the plane have coordinates of the form $(1, \theta, h(\theta))$.

We will now argue that $h(\theta) = \sin(\theta)$. Let θ be given and consider the point $P = (1, \theta, h(\theta))$. The projection of P onto the reference plane is $A = (1, \theta, 0)$ and its projection onto the reference axis is $B = (\cos(\theta), 0, 0)$. The triangle $\triangle PAB$ is isoceles, since the angled plane intersects the cylinder at 45°, thus the line segments AP and BA both are of length $|\sin(\theta)|$. Notice that the sign of $h(\theta)$ is the same as the sign of $\sin(\theta)$. This shows that $h(\theta) = \sin(\theta)$.

As shown above, points on the intersection of the cylinder and the angled plane have cylindrical coordinates of the form $(1, \theta, \sin(\theta))$. Thus, as such a point is moved around the cylinder the angle θ varies through a 360° arc and the height coordinate $\sin(\theta)$ traces a whole period of a sine curve.