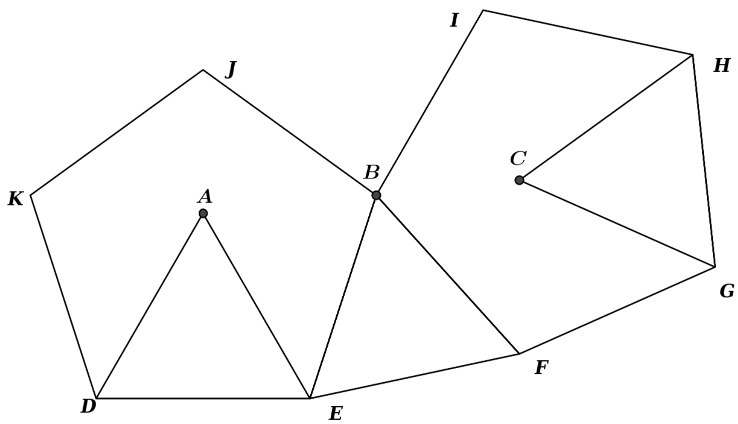
Monthly Mathematics Challenge Solutions – March 2016

1. The triangles and pentagons in the figure are all regular with sides of one unit of length. Show that the points A, B, and C are on a line and find the distance from A to C.



Solution: Refer to the annotated figure above for point labels.

claim 1: $\triangle IHC$ is congruent to $\triangle CFG$. Therefore, $IC = CF$.
proof: $IH = CG = CH = CG = 1$
Since angles in a regular pentagon are equal, $\triangle CHG$ is equilateral. Thus $\angle IHC = \angle IHG - \angle CHG = \angle HGF - \angle HGC = \angle CGF$.

claim 2: $\triangle IBC$ is congruent to $\triangle CBF$. Therefore, $\angle IBC = \angle CBF = \frac{\angle IBF}{2} = 54°$.
proof: $IC = CF = BI = BF = 1$

Since $\triangle BEF$ is equilateral, $\angle EBF = 60°$. Thus, $\angle AEB = \angle BED - \angle AED = 108° - 60° = 48°$.

Since $\triangle ABE$ is isoceles, $\angle ABE = \frac{180° - 48°}{2} = 66°$.

Combining these facts we find that $\angle ABE + \angle EBF + \angle FBC = 66° + 60° + 54° = 180°$. This shows that points A, B, and C lie on a line.

We will now use the law of sines to determine the distance from A to C.

$$\frac{AB}{\sin(48°)} = \frac{1}{\sin(66°)} \text{ implies that } AB = \frac{\sin(48°)}{\sin(66°)}$$

$$\frac{BC}{\sin(42°)} = \frac{1}{\sin(54°)} \text{ implies that } BC = \frac{\sin(42°)}{\sin(54°)}$$

Therefore, $AC = AB + BC = \frac{\sin(48°)}{\sin(66°)} + \frac{\sin(42°)}{\sin(54°)}$.

2. Find the remainder in the division of 3^{2012} by 11.

Solution: We will show that $2012 = 10 \pmod{11}$ and thus the remainder when 2012 is divided by 11 is 10. First note that

$$\begin{aligned}
3^2 &\equiv 9 \equiv -2 \pmod{11} \\
3^3 &\equiv 5 \pmod{11} \\
3^4 &\equiv 15 \equiv 4 \pmod{11} \\
3^5 &\equiv 12 \equiv 1 \pmod{11} \\
3^6 &\equiv 25 \equiv 3 \pmod{11}
\end{aligned}$$

Thus

$$\begin{aligned}
2012 &= 2 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^3 + 3^2 + 3 + 2 \\
&\equiv 6 + 2 + 10 - 2 - 8 - 9 \pmod{11} \\
&\equiv -1 \pmod{11} \\
&\equiv 10 \pmod{11}
\end{aligned}$$

3. Find the smallest number a such that $f(x) = \sin(x) - x + ax^3$ is an increasing function.

Solution: The derivative of $f(x) = \sin(x) - x + ax^3$ is $f'(x) = \cos(x) - 1 + 3ax^2$. Thus, $f(x)$ is increasing when $\cos(x) - 1 + 3ax^2 > 0$ or equivalently when

$$\begin{aligned}
a &> \frac{1 - \cos(x)}{3x^2} \\
&= \frac{2\sin^2(x/2)}{12(x/2)^2} \\
&= \left(\frac{1}{6}\right)\left(\frac{\sin(x/2)}{x/2}\right)^2
\end{aligned}$$

We will now show that we must have $a \geq 1/6$ to insure that $f(x)$ is increasing by showing that $\max\left|\frac{\sin(x)}{x}\right| = 1$. If $x \notin [-1, 1]$, then $\left|\frac{\sin(x)}{x}\right| < 1$. If $x \in [-1, 1]$ and $x \neq 0$, then $|x| > |\sin(x)|$ and $\left|\frac{\sin(x)}{x}\right| < 1$. The limiting value of $\frac{\sin(x)}{x}$ as x tends to zero is 1. This establishes the fact that the smallest number a such that $f(x) = \sin(x) - x + ax^3$ is an increasing function is $a = 1/6$.

4. Let k be a natural number. Show that

$$\gcd\left(\binom{n}{k}, \binom{n+1}{k}, \ldots, \binom{n+k}{k}\right) = 1$$

for any natural number $n \geq k$.

Solution: We will prove this result by induction on k. For $k = 1$, $\binom{n}{k} = n$ and $\binom{n+1}{k} = n+1$. The result follows, since $\gcd(n, n+1) = 1$.

Let $m < n$ be given and assume that the result holds for $k = m$. We will now show that the result holds for $k = m + 1$. By assumption,

$$\gcd\left(\binom{n}{m}, \binom{n+1}{m}, \ldots, \binom{n+m}{m}\right) = 1$$

Let $a_0 = \binom{n}{m+1}$, $a_1 = \binom{n+1}{m+1}$, ..., $a_{m+1} = \binom{n+m+1}{m+1}$, and let $d = \gcd(a_0, a_1, \ldots, a_{m+1})$. Note that we need to show that $d = 1$. By Pascal's identity we have,

$$\binom{b+1}{c+1} = \binom{b}{c+1} + \binom{b}{c}$$

Thus $a_1 = a_0 + \binom{n}{m}$, $a_2 = a_1 + \binom{n+1}{m}$, ..., $a_{m+1} = a_m + \binom{n+m}{m}$ and we see that d must divide each of $\binom{n}{m}, \binom{n+1}{m}, \ldots, \binom{n+m}{m}$. By assumption these terms have $\gcd = 1$. Hence, we must have $d = 1$ and the result is established.

5. Let $z_1 = 1$ and $z_{n+1} = \frac{1}{2}(z_n + \frac{i}{z_n})$ for all $n \geq 1$ (where i is the imaginary unit). Show that $(z_n)_{n=1}^{\infty}$ converges and find its limit.

Solution:

Let $\alpha = 1 + \frac{i}{\sqrt{2}}$. Observe that $\alpha^2 = i$. Now define $w_n = \frac{z_n - \alpha}{z_n + \alpha}$ for all $n = 1, 2, \ldots$. Then

$$w_n^2 = \frac{z_n^2 - 2z_n\alpha + \alpha^2}{z_n^2 + 2z_n\alpha + \alpha^2} = \frac{\frac{1}{2}(z_n + \frac{i}{z_n}) - \alpha}{\frac{1}{2}(z_n + \frac{i}{z_n}) + \alpha} = \frac{z_{n+1} - \alpha}{z_{n+1} + \alpha} = w_{n+1},$$

for all $n = 1, 2, \ldots$. Hence, $w_n = w_1^{2^n}$ for all $n = 1, 2, \ldots$. Observe that

$$|w_1| = \left| \frac{2\sqrt{2}i}{4 + 2\sqrt{2}} \right| < 1.$$

Hence, $w_n \to 0$, i.e., $\frac{z_n - \alpha}{z_n + \alpha} \to 0$. From this we deduce that $z_n \to \alpha$. (We can argue as follows,

$$\frac{z_n - \alpha}{z_n + \alpha} = \frac{z_n - \alpha}{(z_n - \alpha) + 2\alpha} = \frac{1}{1 + \frac{2\alpha}{z_n - \alpha}} \to 0 \Rightarrow z_n - \alpha \to 0.)$$