Mathematics and Statistics Awareness Month 2018

College Level Problems

C1. This question has three related parts:
a) Find all positive integer pairs a and b which satisfy $a^b = b^a$.
b) Does the answer change if we remove the condition that a and b are positive? If it does change, how?
c) What happens if we allow a and b to be rational numbers?

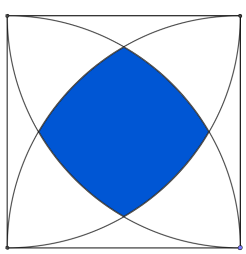
C2. Find all positive integer triples a, b, c which satisfy the equation $a!b! = a! + b! + c^2$

C3. Consider the following powers of 99.

$$99^1 = 99, \qquad 99^2 = 9801, \qquad 99^3 = 970299, \qquad 99^4 = 96059601, \qquad 99^5 = 9509900499$$

Prove that 99^n ends in 99 for every odd natural number n. *Hint* First prove that, if $a \equiv b \bmod k$, then $a^n \equiv b^n \bmod k$.

C4. In the figure below, four quarter circles, connecting diagonally opposite vertices, are drawn in a unit square. Find the area of the region in the center of the square with edges formed by these quarter circles. This region is shaded in the figure below.



C5. Suppose that when the positive integer n is divided by d, the quotient q and the remainder r are such that d, q, and r are consecutive positive integer terms in a geometric sequence, but not necessarily in that order. For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that $4, 6, 9$ are consecutive terms in a geometric sequence with common ratio $3/2$. Prove that n cannot be prime.

C6. Prove that $\tan(a) + \tan(b) + \tan(c) = \tan(a) \cdot \tan(b) \cdot \tan(c)$ for the angles, a, b, c of any non–right triangle.

C7. A bowl contains one red marble and one blue marble. In a game, a player pays \$2 to play and selects a marble at random from the bowl. If the marble is blue, then it is returned to the bowl, one extra red marble is added, and then a marble is selected at random. This game continues until a red marble is selected. Once the game stops the player receives winnings equal in dollars to the number of red marbles that were in the bowl on that particular turn. Find the exact value of the expected return for this game.

C8. Consider an arrangement of n dimes in a straight line. A move consists of taking a dime and turning it over (from head to tail or vice versa) and of doing the same to each of its neighbors. If the dime is at the end of the line, then it will have only one neighbor. For example, if the arrangement is $HHTH$ and you choose the second dime, then your move would result in $TTHH$. The general question is this: Given any arrangement of n dimes, is it possible to find a sequence of moves to bring it to all heads? Your task here is to show that the answer is no for $n = 5$ but yes for $n = 6$.

C9. A rectangular sheet of paper is folded so that two diagonally opposite corners come together. If the crease formed is the same length as the longer side of the sheet, what is the ratio of the longer side of the sheet to the shorter side?

C10. Prove that if the positive integer n that is not a multiple of seven, then its cube is either one more or one less than a multiple of 7.