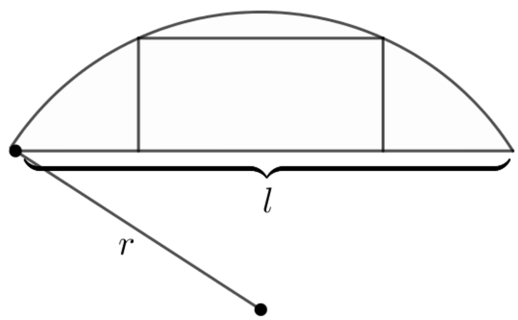
Monthly Mathematics Challenge

University of Louisiana at Lafayette

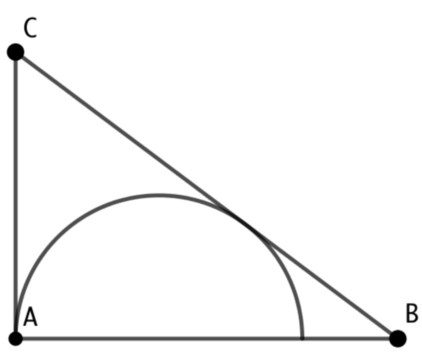
February 2018

Solutions may be submitted at any time. They can be emailed or handed in to Calvin Berry (cberry@louisiana.edu) or Leonel Robert (lrobert@louisiana.edu).

1. Inscribe between a given chord of a circle and the arc of the circle that it subtends the rectangle with greatest area. More specifically, assume that the radius of the circle is r, the length of the chord is $0 \leq l \leq 2r$, and express the lengths of the sides of the rectangle in terms of r and l.



2. The triangle ABC is a right triangle, with the angle at A being the right angle. A semicircle is inscribed as shown. Find the radius of the semicircle in terms of the lengths of the sides of the triangle.



3. A box contains 10 cards numbered 1, 2, 3, 4, and 5 (two cards with each number). Four cards are selected randomly without replacement. What is the probability that 4 is the largest value selected?

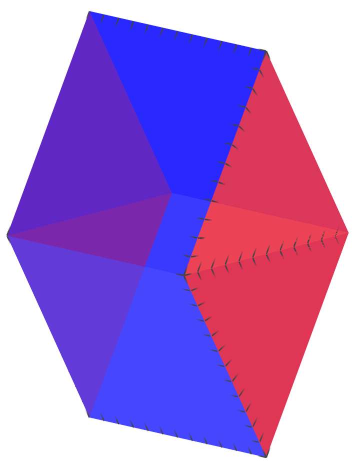
4. Solve the system of ODEs (with arbitrary initial conditions):

$$\begin{aligned}\dot{\phi} &= e^{\phi}\cos(\psi)\\ \dot{\psi} &= -e^{\phi}\sin(\psi).\end{aligned}$$

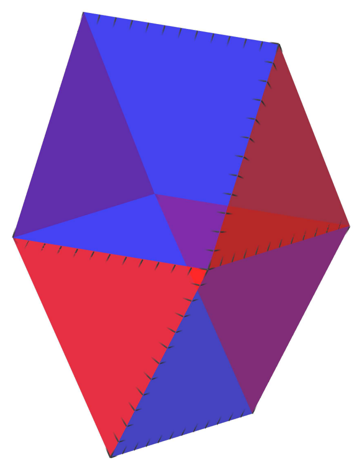
5. Two polyhedra are shown below. Polyhedron (A) is a prism whose "base" is a rhombus made up of two equilateral triangles (front and back faces in the picture) and the other faces are squares. The faces of polyhedron (B) are squares and equilateral triangles.

(a) Show that 3–dimensional space can be filled (without leaving any gaps) with infinitely many identical copies of polyhedron (A).

(b) Show that 3–dimensional space can be filled (without leaving any gaps) with infinitely many identical copies of polyhedron (B).



(A)



(B)

6. Let n and k be positive integers such that $k < n$. Find the number of pairs (x, y) of positive integers that are not larger than n and satisfy the inequality $|x - y| \le k$.

7. Let M be a set of real numbers that satisfies the following conditions:
i) The number 1 is an element of M.
ii) If the number $\sqrt[3]{x}$ is in M, then so is the number $1 + x$.
iii) If the number x is in M then so is $\sqrt{x}$.
Show that the numbers 3 and $2\sqrt{2} + 1$ are elements of M.

8. Find all the numbers a for which the equation $2^{x^2 - ax} - 2^{x-1} + x^2 - (a+1)x + 1 = 0$ has two distinct real solutions.

9. Find the first decimal (tenths part) for every number in the sequence $(x_n)_{n=1}^{\infty}$, where $x_n = \sqrt{n^2 + n + 1}$.