Monthly Mathematics Challenge

University of Louisiana at Lafayette

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Solutions may be submitted at any time. They can be emailed or handed in to Calvin Berry (cberry@louisiana.edu) or Leonel Robert (lrobert@louisiana.edu).

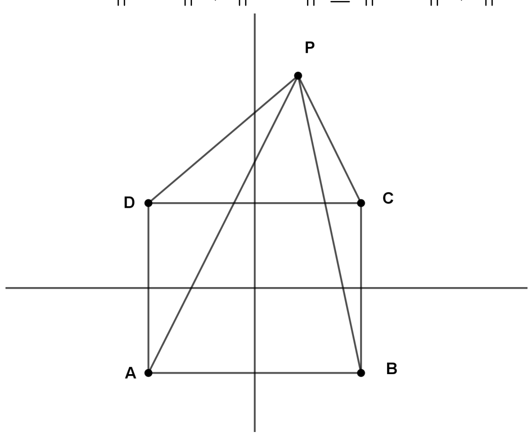
1. Consider a rectangle centered at the origin with corners at points A, B, C, D, as shown in the figure below, and a point P in the first quadrant. For points X and Y, let $\|XY\|$ denote the distance from X to Y, *i.e.*, the length of the line segment XY.

(i) Show that, for any point P in the first quadrant, the sums of the squares of the distances from P to the diagonally opposite corners are equal. That is,

$$\|PB\|^2 + \|PD\|^2 = \|PA\|^2 + \|PC\|^2.$$

(ii) Show that, for any point P in the first quadrant, the sums of the distances from P to the diagonally opposite corners satisfy the following inequality.

$$\|PB\| + \|PD\| \geq \|PA\| + \|PC\|.$$



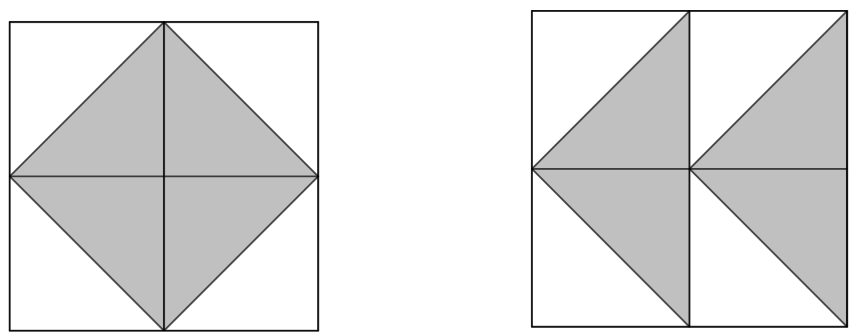
2. Prove that, given any set of $n \geq 2$ distinct vectors in the Euclidean plane, say $v_1, \ldots, v_n$, it is possible to choose scalars, $a_1, \ldots, a_n$, where each $a_i \in \{1, -1\}$ so that

$$\left\| \sum_{i=1}^{n} a_i v_i \right\|^2 \geq \sum_{i=1}^{n} \|v_i\|^2.$$

3. We will say that a sequence $(x_n)_{n=1}^{\infty}$ is *almost increasing* if there is a sequence $(e_n)_{n=1}^{\infty}$ such that $x_{n+1} - x_n \geq -e_n$ for all n and $\sum_{n=1}^{\infty} e_n$ is finite.

Prove that if $(x_n)_{n=1}^{\infty}$ is *almost increasing* and bounded from above, then it is convergent.

4. The large squares shown below consist of 4 squares divided into triangles, of which one is shaded. Each large square has two 1×2 rows and two 2×1 columns. The square at the right can be obtained from the square at the left by "flipping" its right column. That is, by reflecting across a vertical line through the center of the column. If you are allowed to perform any number of such flips of the rows and/or columns of the square on the left, how many different patterns can you create?



5. The Fibonacci numbers form a sequence, the Fibonacci sequence, where each number is the sum of the two preceding it. That is,

$$F_1 = 1, \quad F_2 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3$$

The beginning of the sequence is thus: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$

For $n \geq 2$, find the number of ways in which the n^{th} Fibonacci number F_n can be written as a sum of distinct Fibonacci numbers. For this problem do not use F_1 in the sum. For example:

$F_2 = F_2$

$F_3 = F_3$

$F_4 = F_4$ and $F_4 = F_3 + F_2$

$F_5 = F_5$ and $F_5 = F_4 + F_3$

$F_6 = F_6$ and $F_6 = F_5 + F_4$ and $F_6 = F_5 + F_3 + F_2$