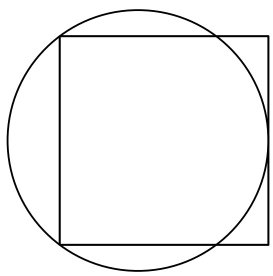
Mathematics and Statistics Awareness Month Problems

High School Problems

B1. What is the smallest integer n greater than 1 such that for any positive integer k, k raised to the power n and k have the same last digit?

B2. Let A be a unit square. What is the largest area of a triangle whose vertices lie on the perimeter of A? Justify your answer.

B3. In the diagram the square has two of its vertices on the circle of radius 1 unit and the other two vertices lie on a tangent to the circle. Find the area of the square.



B4. Two forgetful friends agree to meet in a coffee shop one afternoon but each has forgotten the agreed time. Each remembers that the agreed time was somewhere between 2 pm and 5 pm. Each decides to go to the coffee shop at a random time between 2 pm and 5 pm, wait for half an hour, and leave if the other doesn't arrive.
Find the probability that they meet.

B5. Eight islands each have one or more air services. An air service consists of flights to and from another island, and no two services link the same pair of islands. There are 17 air services in all between the islands.
Show that it must be possible to use these air services to fly between any pair of islands.

B6. Note that $654/545 = 6/5$, so that one can "cancel" the 54 in the numerator and denominator of $654/545$ without changing the value of the fraction. Now consider the fraction $6545454 \cdots 54/545454 \cdots 545$, where there are n copies of 54 following the digit 6 of the numerator, and the same number n of copies of 54 preceding the final digit 5 of the denominator.
Prove that this fraction is equal to $6/5$ for every positive integer n.

B7. Cheryl chose three distinct integers between 1 and 5. We would like to identify these numbers. We are allowed to make queries in the following form: we give Cheryl 3 distinct numbers a, b, and c, and she tells us how many of these numbers are among the chosen ones. For example, if we guess 2, 3, and 4 and the chosen numbers are 1, 2, and 5, we get the answer 1.
Find the smallest number of queries needed to guarantee that we can always identify the three chosen numbers. Note that we do not need to actually name the three correct numbers in a query; it is sufficient to identify them without a doubt from the replies to our queries.

B8. Some people are happy if their nth birthday occurs in a year for which the last two digits are the reverse of the digits of n. For example, someone born in 1945 would be happy on their 38th birthday, which occurs in 1983.
Devise a formula that tells, for any year n (n greater than or equal to 0), how often a person who is born in year n will experience a happy birthday. Assume the person lives to have a 99th birthday, but not a 100th; and also count the true date of birth as the 00th birthday. That is, birthdays run from 00 to 99 (always two digits). Ignore leap days.
In particular, in what year should one be born to maximize happy birthdays?

B9. Let n be an integer whose decimal digits from left to right are nondecreasing (e.g., 113445889) and having distinct digits as its two rightmost digits. Let S be the sum of the digits of $9n$. Then S is divisible by 9, so $S = 9m$.
How large can m be?

B10. An irregularly shaped flat yard is surrounded by a tall fence built from connected straight pieces. We call a location sightful if it is inside the yard and a surveyor (shorter than the fence) can see every part of the inside of the fence from that location. (The surveyor is allowed to turn around on the spot). If A is a sightful location and B is a sightful location, show that every point on the straight path from A to B is sightful. (In the figure the point P is sightful, while the point Q is not.)