Modeling Predator-Prey Systems: Seasonal Breeding and Prey Evolution

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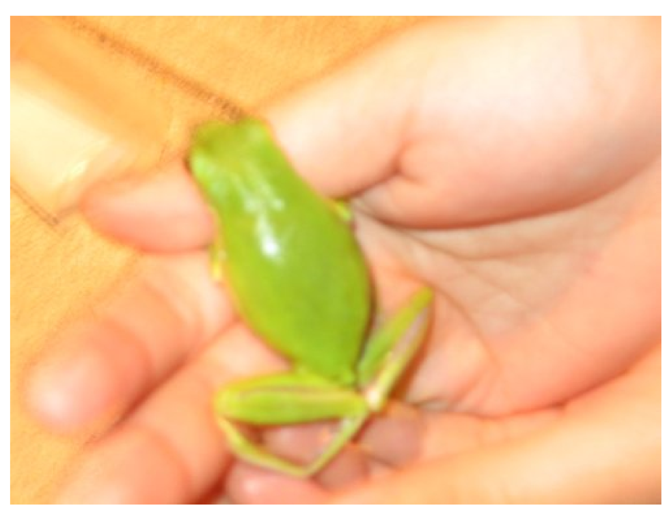
Motivation

a). Ackleh et al.(2019): Persistence and stability analysis of discrete-time predator-prey models: a study of population and evolutionary dynamics.

b). **Previous Project:** Prey reproduction is continuous, meaning it occurs at each time unit, and the prey evolves in response to toxicants.

c). **Current Project:** What happens when prey reproduction is periodic, meaning it occurs at alternating time units, and prey evolves in response to toxicants?

d). Example: **Green Treefrogs**, *Hyla cinerea* - small, arboreal amphibian species with breeding season ≈ 6 months. **(2 periodic birth)**



Source: https://userweb.ucs.louisiana.edu/ asa5773/ubm/gallery-2006.html

The periodic model and its composite map

The periodic model:

$$n(t+1) = R(t,n(t),p(t),v)\, n(t)\big|_{v=u(t)},$$
$$p(t+1) = [s_p p(t) + \kappa\hat{\phi}(t,n(t),v)n(t)\, f(p(t),v)p(t)]\big|_{v=u(t)},$$
$$u(t+1) = u(t) + \nu\partial_v \ln[R(t,n(t),p(t),v)]\big|_{v=u(t)}.$$

Time transformation: $\tau + i = 2(t+i); \quad i \geq 0 \implies \tau = 2t, \tau+1 = 2t+2, \ldots$

The composite map:

$$n(\tau+1) = \hat{R}(n(\tau),p(\tau),v)n(\tau)\big|_{v=u(\tau)},$$
$$p(\tau+1) = [s_p x + \kappa n AB f(x,y)x]\big|_{v=u(\tau)},$$
$$u(\tau+1) = u(\tau) + \nu\partial_v \ln \hat{R}(n(\tau),p(\tau),v)\big|_{v=u(\tau)}, \quad \text{where}$$

$\hat{R}(n,p,u) := ABC,\ A(n,u) := s_n^2 + s_n\hat{b}(n,u)[1-\epsilon(u)],\ B(p,u) := 1 - f(p,u)p,$
$C(n,p,u) := 1 - f(x,y)x,\ \ x(n,p,u) := p(2t+1) = s_p p + (\kappa/s_n)Anfp,$
$y(n,p,u) := u(2t+1) = u + \nu\partial_u \ln[(AB/s_n)]$

The composite model is much more complex than the model with continuous (prey) reproduction.

The growth rates, and toxicant thresholds

Inherent growth rates: $r_0^P := s_n^2 + s_n b(0)[1-\epsilon(0)], \quad \tilde{r}_0^P := s_n^2 + s_n b(\tilde{u})[1-\epsilon(\tilde{u})].$

Invasion growth rates: $r_i^P := s_p^2 + \kappa\left(\frac{s_p}{s_n} + s_p + \frac{\kappa\bar{n}}{s_n}f(0,0)\right)\bar{n}f(0,0),$

$\tilde{r}_i^P := s_p^2 + \kappa\left(\frac{s_p}{s_n} + s_p + \frac{\kappa\tilde{n}}{s_n}f(0,\tilde{u})\right)\tilde{n}f(0,\tilde{u}).$

Toxicant thresholds: $\epsilon_{c_1} = \dfrac{b_1}{b_1+\epsilon_1}, \quad \epsilon_{c_2}, \quad \epsilon_{c_3}.$

Theoretical results from the composite model

a). The extinction equilibrium $(0,0,0)$ is locally asymptotically stable if $r_0^P < 1, \epsilon_0 < \epsilon_{c_1}$ and $0 < \nu < \frac{s_n + b_0(1-\epsilon_0)}{b_0\{b_1(1-\epsilon_0)-\epsilon_0\epsilon_1\}}$, and unstable otherwise.

b). The predator-free equilibrium $(\bar{n},0,0)$ exists if $r_0^P > 1$ and is locally asymptotically stable if $\epsilon_0 < \epsilon_{c_1}, r_i^P < 1$ and $0 < \nu < \frac{s_n + b_0\hat{b}(\bar{n})(1-\epsilon_0)}{b_0\hat{b}(\bar{n})\{b_1(1-\epsilon_0)-\epsilon_0\epsilon_1\}}$.

c). The equilibrium $(n^*,p^*,0)$ with $n^*,p^* > 0$, exists if $r_0^P, r_i^P > 1$. Moreover, it is locally asymptotically stable if $\epsilon_0 < \epsilon_{c_2}$, and $0 < \nu < \nu^*$.

d). The equilibrium $(0,0,\tilde{u})$ exists if $\epsilon_0 > \epsilon_{c_1}$, and is locally asymptotically stable if $\tilde{r}_0^P < 1$, and $-2 < \nu\frac{\partial^2(\ln\hat{R})}{\partial v^2}\big|_{(0,0,\tilde{u})} < 0$.

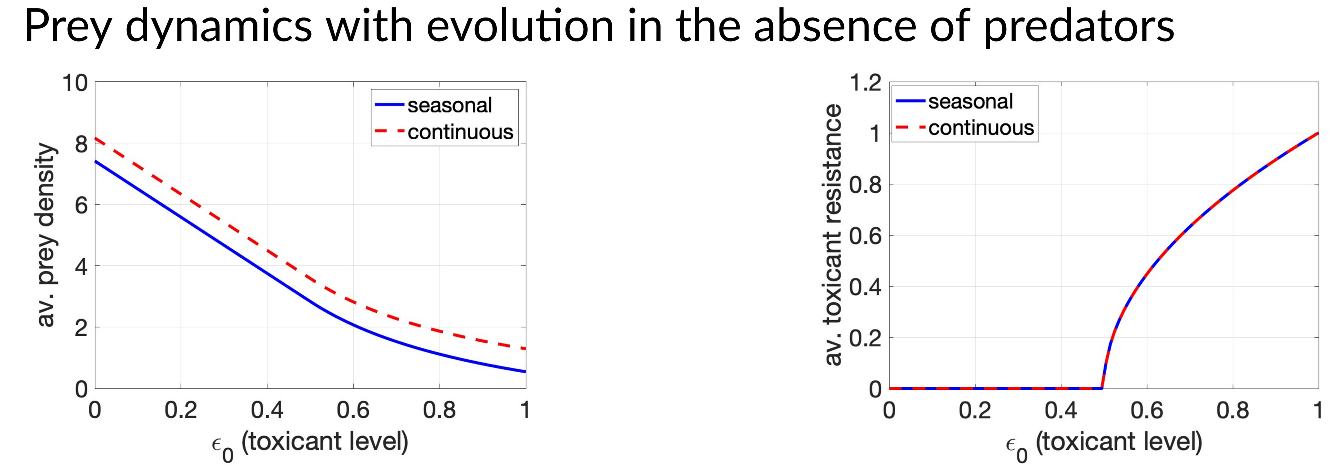
e). The equilibrium $(\tilde{n},0,\tilde{u})$ exists if $\epsilon_0 > \epsilon_{c_1}$, and $\tilde{r}_0^P > 1$, and is locally asymptotically stable if $\tilde{r}_i^P < 1$, and $-2 < \nu\frac{\partial^2(\ln\hat{R})}{\partial v^2}\big|_{(\tilde{n},0,\tilde{u})} < 0$.

f). If $\epsilon_0 > \epsilon_{c_3}, \tilde{r}_0^P > 1$, and $\tilde{r}_i^P > 1$, then the evolutionary model is persistent, that is, $\exists\, \epsilon > 0$ such that $\liminf_{\tau\to\infty} min(n(\tau),p(\tau),u(\tau)) > \epsilon$ for any initial condition $n(0),p(0),u(0) > 0$.

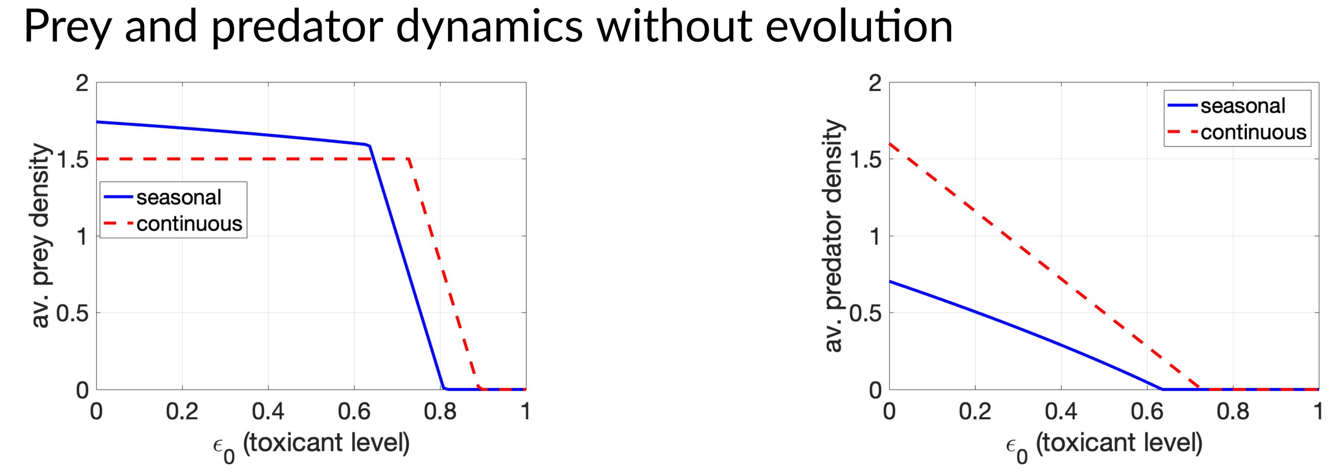
These results suggest that equilibrium stability, long-term persistence, and coexistence depend on specific parameter choices.

Numerical results

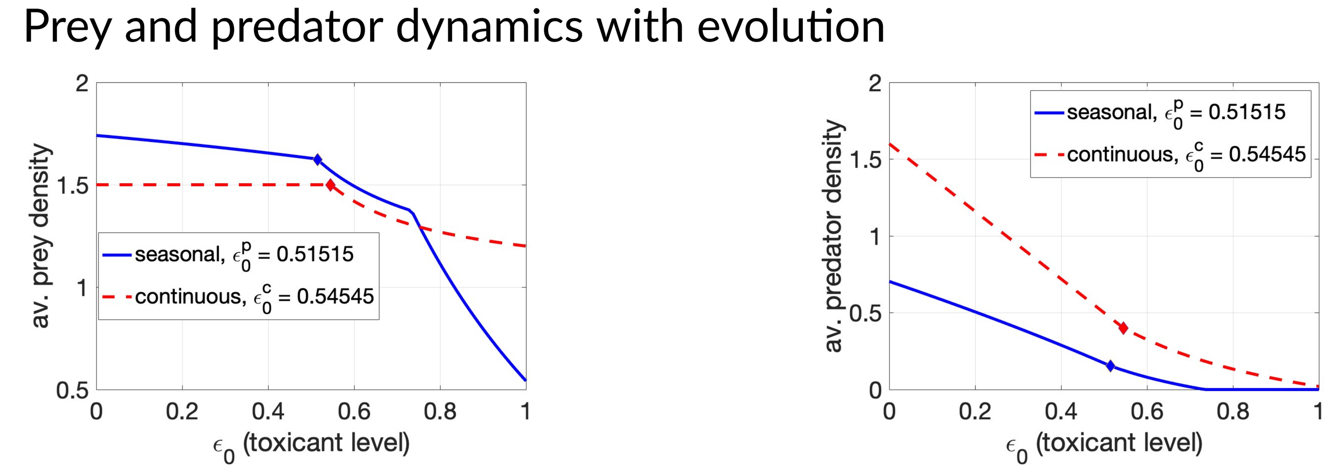
a). Prey dynamics with evolution in the absence of predators



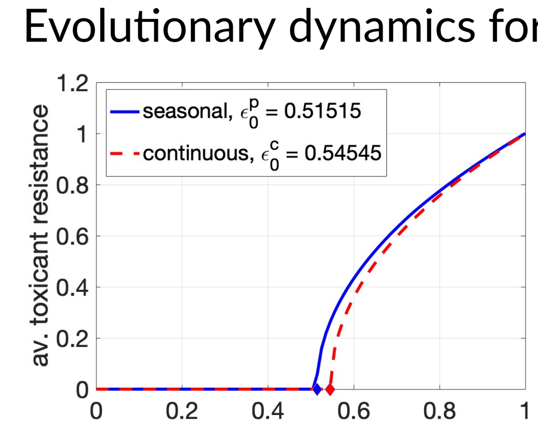
b). Prey and predator dynamics without evolution



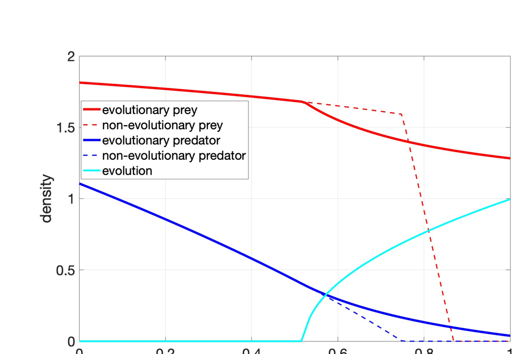
c). Prey and predator dynamics with evolution



d). Evolutionary dynamics for case c)



e). Species with and without evolution



Summary of the dynamics of the periodic model

Steady States	Existence Conditions	LAS Conditions
Extinction Equilibrium, $u=0$	- -	$r_0^P < 1, \epsilon_0 < \epsilon_{c_1}, 0 < \nu < \frac{s_n+b_0(1-\epsilon_0)}{b_0\{b_1(1-\epsilon_0)-\epsilon_0\epsilon_1\}}$
Predator-free 2-cycle, $u=0$	$r_0^P > 1$	$r_i^P < 1, \epsilon_0 < \epsilon_{c_1}, 0 < \nu < \frac{s_n+b_0\hat{b}(\bar{n})(1-\epsilon_0)}{b_0\hat{b}(\bar{n})\{b_1(1-\epsilon_0)-\epsilon_0\epsilon_1\}}$
Predator-prey 2-cycle, $u=0$	$r_0^P > 1, r_i^P > 1$	$\epsilon_0 < \epsilon_{c_2}$, and $\exists\, \nu^* > 0$ for all $0 < \nu < \nu^*$
Extinction Equilibrium, $u>0$	$\epsilon_0 > \epsilon_{c_1}$	$\tilde{r}_0^P < 1, -2 < \nu\frac{\partial^2(\ln\hat{R})}{\partial v^2}\big\|_{(0,0,\tilde{u})} < 0$
Predator-free 2-cycle, $u>0$	$\epsilon_0 > \epsilon_{c_1}, \tilde{r}_0^P > 1$	$\tilde{r}_i^P < 1, -2 < \nu\frac{\partial^2(\ln\hat{R})}{\partial v^2}\big\|_{(\tilde{n},0,\tilde{u})} < 0$
Predator-prey 2-cycle, $u>0$	$\epsilon_0 > \epsilon_{c_3}, \tilde{r}_0^P > 1, \tilde{r}_i^P > 1$	$\exists\, \hat{\nu} > 0$ for all $0 < \nu < \hat{\nu}$

Table 1. Summary of dynamics; seasonal model. LAS abbreviates Locally Asymptotically Stable

Concluding remarks

a). Seasonality is always deleterious for a single species in a toxic environment, in agreement with the Cushing-Henson conjectures.

b). While seasonality is advantageous at low and intermediate toxicant levels, it can be deleterious at high levels when prey do not evolve.

c). When predators are present, the evolving prey follows a pattern similar to the case of nonevolving prey but with different dynamics; namely the prey is able to benefit more from seasonal than continuous reproduction by evolving its trait value.

d). Whether the prey is evolving or not, seasonality is always detrimental to the predator.

e). The inclusion of evolution helps both species maintain a higher density.

Future work

a). The impact of evolution on either predator or prey when predator or prey is structured.

b). The impact of seasonality on frequency-dependent evolution in a predator-prey system.

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References

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[2] A.S. Ackleh, N. Pant and A. Veprauskas, A Discrete-Time Predator-Prey Model with Seasonal Breeding, International Conference on Difference Equations and Applications, Springer, 2023, pp. 233–257.