Discrete-Time Refuge-Mediated Selection Model

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Motivation

- In [1] the authors examine the impact of a safe refuge on a continuous-time selection model with multiple (n) traits. The model assumes that if individuals with 'weak' traits lack a safe refuge they will go extinct, implying that $n-1$ safe refuge patches are needed for the survival of n traits.
- In this work, we extend the continuous Nonlinear Refuge-Mediated Selection Model developed by Ackleh et al. in [1] into a discrete-time model and present a numerical analysis as well as an analytical examination of two species stability and persistence. We present a competitive exclusion analysis without dispersal and set up a framework for a stability analysis of the full discrete model with dispersal, specifically for the case of two species (one with a refuge).

The Model

$$x_i(t+1) = \frac{a_i}{1+b_i f(X(t))}x_i(t) + s_i(1-\delta_i)x_i(t) + \tilde{s}_i\epsilon_i y_i(t), \quad i=1,...,n-1$$
$$y_i(t+1) = \frac{r_i}{1+c_i g_i(y_i(t))}y_i(t) + \tilde{s}_i(1-\epsilon_i)y_i(t) + s_i\delta_i x_i(t), \quad i=1,...,n-1$$
$$x_n(t+1) = \frac{a_n}{1+b_n f(X(t))}x_n(t) + s_n x_n(t)$$
$$X(t) = \sum_{j=1}^{n}\theta_j x_j(t)$$

$x_i(t)$= individuals carrying the i-th trait in the competition patch at time t
$y_i(t)$= individuals carrying the i-the trait in the i-th safe refuge patch at time t.
$x_n(t)$= individuals carrying the n-th (fittest) trait.
The function $X(t)=\sum_{j=1}^{n}\theta_j x_j(t)$= weighted population size in the competition patch.
a_i, r_i=the intrinsic growth rate of an individual carrying the i-th trait in the competition patch and in the safe-refuge patch (respectively).
b_i, c_i= competition coefficient in the competition and each safe refuge patch (respectively).
$s_i, \tilde{s}_i$= survival probability of the i-th trait in the competition patch and in the safe-refuge patch (respectively).
δ_i=dispersal from the competition patch to the refuge patch.
ϵ_i=dispersal from the refuge patch to the competition patch.

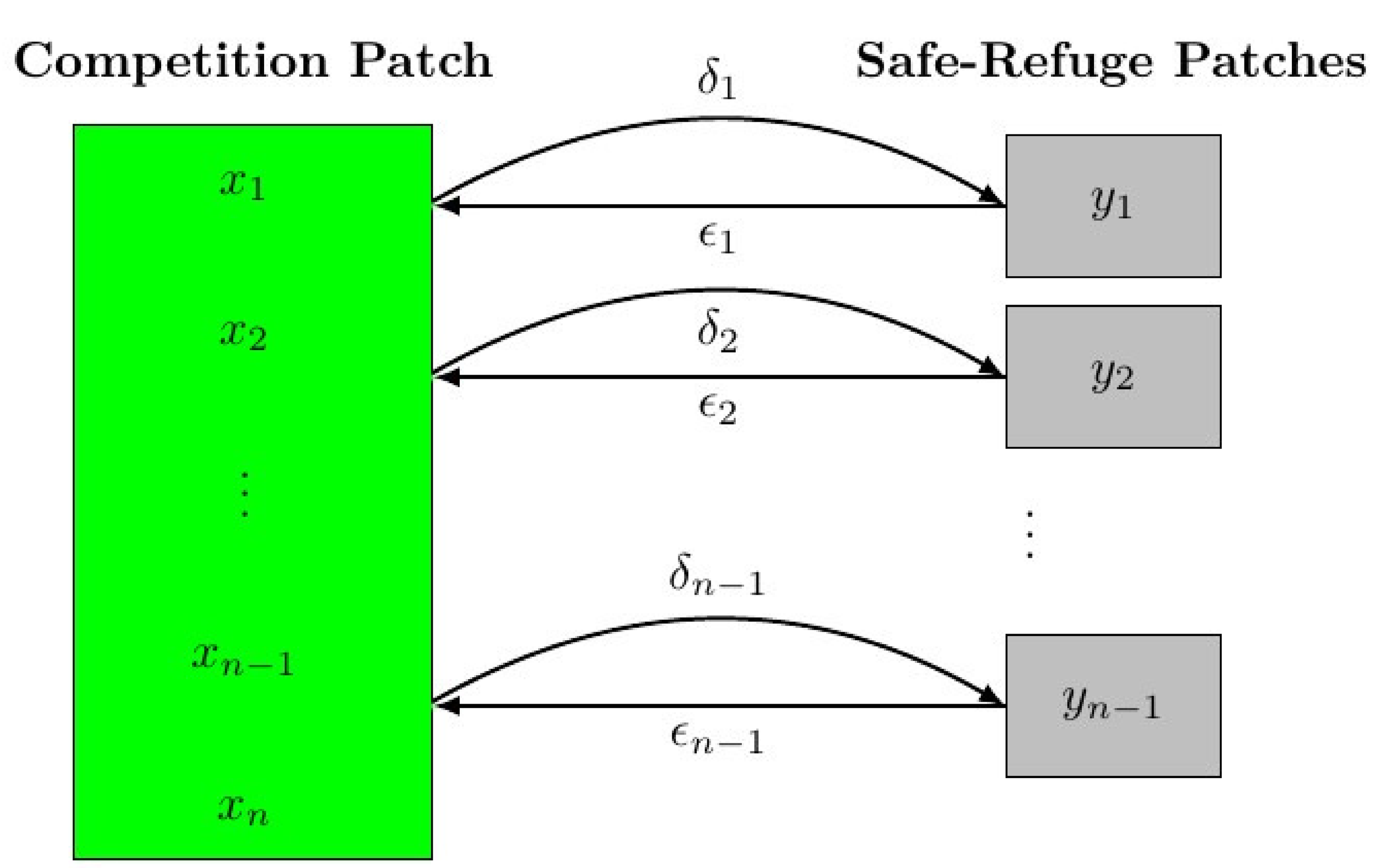

Figure 1. Dynamics between the patches

Absence of Dispersal

$$x_i(t+1) = \frac{a_i}{1+b_i f(X(t))}x_i(t) + s_i x_i(t), \quad i=1,2,...,n \tag{1}$$

- When $\frac{a_n+s_n-1}{b_n(1-s_n)} \geq \frac{a_i+s_i-1}{b_i(1-s_i)}\ \forall i=1,\dots,n-1$, the equilibrium $\left(0,\dots,0,\frac{a_n+s_n-1}{b_n(1-s_n)}\right)$ is asymptotically stable and attracts all solutions $x(t)$ of (1) with $x_n(0)>0$.

Important Result: Competitive Exclusion

The above theorem proves that the species with the fittest trait wins the competition by driving all other species to extinction.

A Two Species Model

$$x_1(t+1) = \frac{a_1}{1+b_1(x_1(t)+x_2(t))}x_1(t) + s_1(1-\delta_1)x_1(t) + \tilde{s}_1\epsilon_1 y_1(t)$$
$$y_1(t+1) = \frac{r_1}{1+c_1 y_1(t)}y_1(t) + \tilde{s}_1(1-\epsilon_1)y_1(t) + s_1\delta_1 x_1(t) \tag{2}$$
$$x_2(t+1) = \frac{a_2}{1+b_2(x_1(t)+x_2(t))}x_2(t) + s_2 x_2(t)$$

- $\lambda_0 = \frac{1}{2}\left(a_1+s_1(1-\delta_1)+r_1+\tilde{s}_1(1-\epsilon_1)\right) + \frac{1}{2}\sqrt{[(a_1+s_1(1-\delta_1))-(r_1+\tilde{s}_1(1-\epsilon_1))]^2+4s_1\tilde{s}_1\epsilon_1\delta_1}$ is the inherent growth rate of the weaker species.
- If $\lambda_0<1$, $a_2+s_2<1$, then $E_0=(0,0,0)$ is globally asymptotically stable in $\mathbb{R}^3_+$.
- If $\lambda_0<1$, $a_2+s_2>1$ then $E_0=(0,0,0)$ is unstable and $E_1(0,0,\bar{x}_2)$ is globally asymptotically stable in $\mathbb{R}^3_+\setminus\{(x_1,y_1,0): x_1\geq 0, y_1\geq 0\}$.
- If $\lambda_0>1$, $a_2+s_2<1$, then $E_+=(\bar{x}_1,\bar{y}_1,0)$ is globally asymptotically stable in $\{(x_1,y_1,x_2)\in\mathbb{R}^3_+ | x_1>0, y_1>0\}$.

Two Species Persistence Results

- If $a_2+s_2>1$, then there exists a $\mu_1>0$ such that
$$\liminf_{t\to\infty} x_2(t) > \mu_1,$$
for all initial conditions satisfying $x_2(0)>0$.
- If $\lambda_0>1$ then there exists a $\mu_2>0$ such that
$$\liminf_{t\to\infty} min(x_1(t),y_1(t)) > \mu_2,$$
for all initial conditions satisfying $x_1(0)+y_1(0)>0$.
- If $\lambda_0>1$ and $a_2+s_2>1$, then there exists a $\mu>0$ such that
$$\liminf_{t\to\infty} min(x_1(t),y_1(t),x_2(t)) > \mu, \tag{3}$$
for all initial conditions satisfying $x_1(0)+y_1(0)>0, x_2(0)>0$.

Biological Interpretation of Two Species Persistence

- If the inherent growth rate of species 1 is high enough ($\lambda_0>1$) then species 1 will persist over time given that it has a feasible initial condition $(x_1(0)+y_1(0)>0)$.
- If the combined effects of the intrinsic growth rate (a_2) and survival (s_2) for species 2 are strong enough, ($a_2+s_2>1$) then species 2 will persist over time given that it has a feasible initial condition $(x_2(0)>0)$.
- If both above conditions are met then both species will persist over time.

Numerical Results

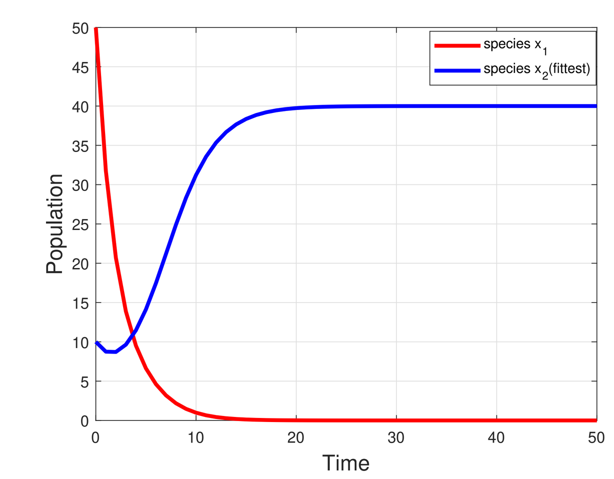

Figure 2. When there is no dispersal the fittest species (x_2) drives the weaker species (x_1) in the competition patch to extinction.

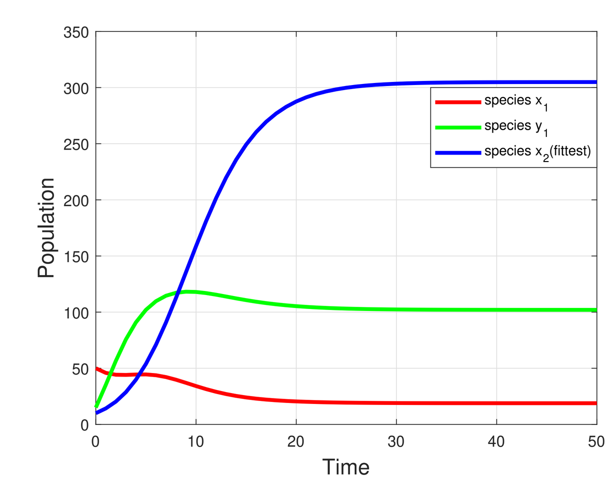

Figure 3. All species coexist because of dispersal.

Conclusion

- We have shown that in the absence of dispersal, the species having the strongest trait outcompetes the weaker species and drives the weaker species to extinction.
- When dispersal is present, we were able to prove that all species, provided that each weak species has a safe refuge to disperse to, can survive. We proved this by showing uniform persistence of the system with two species.

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References

[1] Azmy S Ackleh, Youssef M Dib, and SR-J Jang.
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